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ON THE DIRECT SOLUTION OF THE GOVERNING EQUATION

FOR RADIATION-RESISTED SHOCK WAVES

By Walter E. Pearson

SUMMARY

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This investigation is concerned with the singular, nonlinear, integral equation that arises in the theoretical study of inviscid, radiation-resisted shock waves without heat conduction. A simple, iterative technique is given for the solution of this equation. Sufficient conditions for the convergence of the iteration are given, and the technique is then applied to two examples with different shock and radiation strengths. When the velocity profiles obtained are compared with earlier results which adapt a specialized approximation from astrophysics, very close agreement is revealed.

AUTHOR

INTRODUCTION

Recent studies (refs. 1 and 2) of radiation-resisted shock waves have exhibited velocity and temperature profiles for characteristic combinations of shock strength and radiation intensity. The singular, nonlinear integral equation which governs the physical structure of the shock wave is developed from the laws of conservation of mass, momentum, and energy; its solution is then approximated to obtain the profiles mentioned.

The integrand of the governing integral equation contains the influence function

$$K(y,z) = \operatorname{sgn}(y - z)E_2(|y - z|)$$

where $E_2(t)$ is the integro-exponential function of order 2. (See ref. 3.) References 1 and 2 approximate this influence function by the use of

$$E_2(t) \approx me^{-nt}$$

The constants, m and n, are assigned by criteria which are intuitively chosen to fit the physical situation.

This kind of approximation has long been used in the radiation problems of astrophysics. It is natural that the question of its utility in the realm of gas dynamics should arise. The present study was initiated to provide an answer to this question, and it uses the analysis leading to the governing equation, developed in the earlier papers, as its starting point. During the course of the investigation, it was realized that the technique of successive approximations was not only applicable for purposes of the present study, but obviates the use of more intricate methods in many problems of radiative transfer.

It is therefore intended that the present paper serve two functions: first, to present a direct method of solving the governing equation, and second, to compare results from the direct solution and the approximate solution obtained in reference 1.

IMPORTANT SYMBOLS

$E_2(t)$	integro-exponential function of order 2
h	specific enthalpy
$K(y,z)$	kernel fuction of the governing integral equation
p	static pressure
q	heat flux due to radiation
T	gas temperature
$u(y)$	gas velocity
$v(y)$	normalized gas velocity
y	"dimensionless distance" from the shock wave
γ	ratio of specific heats
λ	parameter of the governing integral equation
ρ	gas density
$\psi(y)$	the integral appearing in the governing equation, multiplied by λ

ANALYSIS

A brief recounting of the basic equations of references 1 and 2 will be given to establish the point of departure for the present work.

The Governing Equation

The flow under consideration is one-dimensional, steady, inviscid, and without heat conduction, but the gas is both a radiating and absorbing medium.

The conservation laws then give

$$\frac{d}{dx}(\rho u) = 0 \qquad \text{(mass)} \tag{1a}$$

$$\frac{dp}{dx} + \rho u \frac{du}{dx} = 0 \qquad \text{(momentum)} \tag{1b}$$

$$\rho u \frac{dh}{dx} - u \frac{dp}{dx} + \frac{dq}{dx} = 0 \qquad \text{(energy)} \tag{1c}$$

Here, u is the gas velocity, h is the specific enthalpy, p is the static pressure, ρ is the density, and q is the heat flux due to radiation. These equations can be integrated immediately to give

$$\rho u = \Gamma \tag{2a}$$

$$\Gamma u + p = \Gamma c_1 \tag{2b}$$

$$\Gamma[h + (1/2)u^2] + q = \Gamma c_2 \tag{2c}$$

where Γ, c_1, and c_2 are constants of integration.

At this point, it is convenient to assume that the gas is a perfect gas with gas constant $R = c_p - c_v$, and a constant ratio of specific heats $\gamma = c_p/c_v$. The perfect gas assumption also implies the relations

$$p = \rho RT \quad \text{and} \quad h = c_p T$$

where T is the gas temperature. These relations, when combined with equations (2a), (2b), and (2c), give

$$T = \frac{u(c_1 - u)}{R} \tag{3}$$

and

$$u^2 - \frac{2\gamma}{\gamma + 1} c_1 u + \frac{2(\gamma - 1)}{\gamma + 1} c_2 = \frac{2(\gamma - 1)}{\Gamma(\gamma + 1)} q \tag{4}$$

Equation (4) is the governing equation for the velocity profile, and equation (3) defines the temperature profile in terms of velocity. It remains to supply an expression for the radiative heat flux, q.

In reference 1, q is defined in terms of a dimensionless distance, y, given by

$$y = \int_0^x \alpha(z)dz$$

where the function $\alpha(x)$ is the absorption coefficient for an absorbing and emitting grey gas. The expression for q is

$$q(y) = 2\int_{-\infty}^{y} \sigma T^4(z)E_2(y - z)dz - 2\int_{y}^{\infty} \sigma T^4(z)E_2(z - y)dz \tag{5}$$

In this equation, σ is the Stefan-Boltzmann constant, and $E_2(t)$ is a particular member of the family of integro-differential functions of order n.

$$E_n(t) = \int_{1}^{\infty} \frac{e^{-tz}}{z^n} dz \tag{6}$$

The expression for $q(y)$ can be shortened conveniently by the convention

$$K(y,z) = \operatorname{sgn}(y - z)E_2(|y - z|) \tag{7}$$

With this notation, equation (5) becomes

$$q(y) = 2\sigma\int_{-\infty}^{\infty} T^4(z)K(y,z)dz$$

and the governing equation (4) is written as

$$u^2(y) - \frac{2\gamma}{\gamma + 1} c_1 u(y) + \frac{2(\gamma - 1)}{\gamma + 1} c_2 = \frac{4(\gamma - 1)\sigma}{\Gamma(\gamma + 1)}\int_{-\infty}^{\infty} T^4(z)K(y,z)dz \tag{8}$$

Now equation (3) shows that T^4 can be expressed in terms of u, so that equation (8) is an integral equation. Before a substitution is made for T^4, however, the variable is changed to

$$v(y) = \frac{u(y)}{c_1}$$

This will further simplify equation (8), and restrict the range of the unknown function in equation (8) to $0 < v(y) < 1$. With this substitution, equation (3) becomes

$$T(y) = \frac{c_1^2}{R}[v(y) - v^2(y)] \tag{9}$$

and equation (8) becomes

$$v^2(y) - \left(\frac{2\gamma}{\gamma + 1}\right) v(y) + \frac{2(\gamma - 1)}{\gamma + 1} \frac{c_2}{c_1^2} = \frac{4(\gamma - 1)\sigma c_1^6}{\Gamma(\gamma + 1)R^4}\int_{-\infty}^{\infty}[v(z) - v^2(z)]^4 K(y,z)dz \tag{10}$$

Further simplification of equation (10) is possible by setting

$$v(-\infty) = v_1$$

$$v(+\infty) = v_2$$

Since $E_2|\pm\infty - z| = 0$ for all z, it follows that $K(\pm\infty,z) \equiv 0$. Hence, v_1 and v_2 are roots of the equation

$$v^2(y) - \left(\frac{2\gamma}{\gamma + 1}\right) v(y) + \frac{2(\gamma - 1)}{\gamma + 1} \frac{c_2}{c_1^2} = 0$$

This shows that

$$\frac{2\gamma}{\gamma + 1} = v_1 + v_2 \quad \text{and} \quad \frac{2(\gamma - 1)}{\gamma + 1} \frac{c_2}{c_1^2} = v_1 v_2$$

Therefore, equation (10) is rewritten as

$$[v(y) - v_1][v(y) - v_2] = \lambda \int_{-\infty}^{\infty} [v(z) - v^2(z)]^4 K(y,z)dz \tag{11}$$

with

$$\lambda = \frac{4(\gamma - 1)\sigma c_1^6}{\Gamma(\gamma + 1)R^4}$$

It is at this point that references 1 and 2 make the approximation

$$E_2(t) \approx me^{-nt}$$

Direct Solution of the Nonlinear Equation

In what is to follow, the notation $\psi(y)$ will be used for the right-hand term of equation (11); that is,

$$\lambda \int_{-\infty}^{\infty} [v(z) - v^2(z)]^4 K(y,z)dz = \psi(y) = -[v_1 - v(y)][v(y) - v_2] \tag{12}$$

Since $v_1 \geq v(y) \geq v_2$ for all y (ref. 1), the right side of equation (12) shows that $\psi(y) \leq 0$ for all y. From

$$v(y) = \frac{v_1 + v_2}{2} \pm \sqrt{\left(\frac{v_1 - v_2}{2}\right)^2 + \psi(y)} \tag{13}$$

it is clear that

$$\psi(y) \geq -\left(\frac{v_1 - v_2}{2}\right)^2$$

Therefore, the range of $\psi(y)$ is bounded by

$$-\left(\frac{v_1 - v_2}{2}\right)^2 \leq \psi(y) \leq 0$$

In reference 1 it is shown that $v(y)$ is a monotone decreasing function, and this provides a means for choosing the proper sign (±) in equation (13). The origin has been fixed so that

$$v(y) = \frac{v_1 + v_2}{2} \qquad \text{for} \quad y = 0$$

and, hence,

$$v(y) \geq \frac{v_1 + v_2}{2} \qquad \text{for} \quad y < 0$$

$$v(y) \leq \frac{v_1 + v_2}{2} \qquad \text{for} \quad y > 0$$

With this in mind, equation (13) can be written

$$v(y) = \frac{v_1 + v_2}{2} - \operatorname{sgn}(y)\sqrt{\left(\frac{v_1 - v_2}{2}\right)^2 + \psi(y)} \tag{14}$$

Equation (14), by removing the ambiguity of sign in equation (13), permits a reformulation of the governing equation (11) with $\psi(y)$ as the unknown function. When $\psi(y)$ is determined, $v(y)$ is uniquely defined. Moreover, the definition of $\psi(y)$ implies that it is a continuous function, defined for all y, and from this fact, it can be shown that $f[\psi(y)]$, where

$$[v(y) - v^2(y)]^4 = f[\psi(y)] = \left\{v_1 v_2 - \psi(y) + [1 - (v_1 + v_2)]\left[\frac{v_1 + v_2}{2} - \operatorname{sgn}(y)\sqrt{\left(\frac{v_1 - v_2}{2}\right)^2 + \psi(y)}\right]\right\} \tag{15}$$

is continuous except for a possible jump discontinuity at $y = 0$.

To see this, it is only necessary to show that $v(y)$ is continuous except for a possible jump discontinuity at $y = 0$.

Since $v(y)$ always takes on the values v_1 and v_2, the right side of equation (12) implies that $\psi(y)$ always takes on the value

$$\psi(\pm\infty) = 0$$

If $\psi(y)$ assumes any value, say $\psi_0 < 0$, then $\psi(y)$ must take on all values between ψ_0 and $\psi(\pm\infty)$ because $\psi(y)$ is a continuous function. Equation (14) then shows that $v(y)$ takes on all values in the two branches

$$v_1 \geq v(y) \geq \frac{v_1 + v_2}{2} + \sqrt{\left(\frac{v_1 - v_2}{2}\right)^2 + \psi_0} \text{ and } \frac{v_1 + v_2}{2} - \sqrt{\left(\frac{v_1 - v_2}{2}\right)^2 + \psi_0} \geq v(y) \geq v_2$$

If $\psi(y)$ takes on the value $-[(v_1 - v_2)/2]^2$, then the two branches become contiguous, and $v(y)$ is a continuous function. If $\psi(y)$ does not take on this value, $v(y)$ makes a jump at $y = 0$ which is symmetrical about the ordinate $(v_1 + v_2)/2$.

With $\psi(y)$ as defined in equation (12), and $f[\psi(y)]$ as defined in equation (15), the governing equation (11) can be rewritten

$$\psi(y) = \lambda \int_{-\infty}^{\infty} f[\psi(z)]K(y,z)dz \tag{16}$$

Equation (16) will be solved for $\psi(y)$, which will then yield the velocity profile, $v(y)$, through equation (14). However, to solve equation (16), it will be convenient to rid the integral of its infinite limits.

That this is possible can be seen by referring back to equation (11) and writing

$$\int_{-\infty}^{\infty}[v(z) - v^2(z)]^4 K(y,z)dz = \int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{\infty} \tag{17}$$

where a is a constant to be determined later. It can be shown that for any positive number, R, it is possible to choose the constant, a, so that

$$\left|\int_{-\infty}^{-a}[v(z) - v^2(z)]^4 K(y,z)dz\right| + \left|\int_{a}^{\infty}[v(z) - v^2(z)]^4 K(y,z)dz\right| < 10^{-R}$$

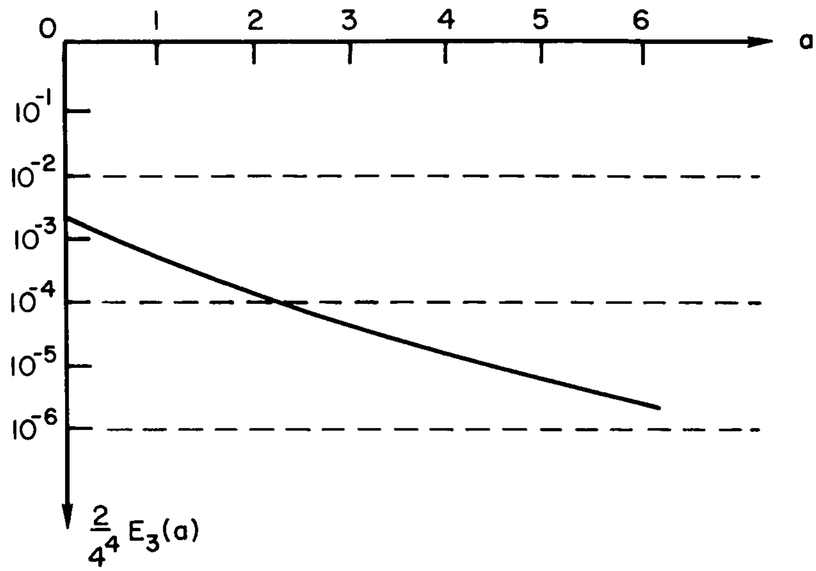
In fact, if an error no larger than 10^{-R} is permissible in $v(y)$, then it suffices that a be chosen so that

$$10^{-R} > \frac{2}{4^4} E_3(a)$$

Since $E_3(t)$ is monotonic decreasing with

$$\lim_{t\to\infty} E_3(t) = 0$$

the constant, a, can always be determined. Sketch (a) shows the relationship between $(2/4^4)E_3(a)$ and a.



Sketch (a)

With a chosen to provide an error no larger than 10^{-R} in $v(y)$, interest is centered on the interval

$$-a \leq y \leq a$$

for it is clear that

$$v(y) \geq v_1 - 10^{-R} \qquad \text{if} \quad y < -a$$

$$v(y) \leq v_2 + 10^{-R} \qquad \text{if} \quad y > a$$

For y in the interval [-a, a], and a permissible error of 10^{-R} in $v(y)$, equation (16) is replaced by

$$\psi(y) = \lambda \int_{-\infty}^{-a} (v_1 - v_1^2)^4 K(y,z)dz + \lambda \int_{-a}^{a} f[\psi(z)]K(y,z)dz + \lambda \int_{a}^{\infty} (v_2 - v_2^2)^4 K(y,z)dz \tag{18}$$

Hence, if $\int_{-\infty}^{-a}$ and $\int_{a}^{\infty}$ are carried out (ref. 3), the governing equation (11) is written

$$\psi(y) = T(y) + \lambda \int_{-a}^{a} f[\psi(z)]K(y,z)dz \tag{19}$$

where

$$T(y) = \lambda(v_1 - v_1^2)^4 E_3(y + a) - \lambda(v_2 - v_2^2)^4 E_3(a - y) \tag{20}$$

Equation (19) is solved by the construction of a sequence of functions $\left\{\psi_n(y)\right\}$ defined as follows:

$$\psi_{n+1}(y) = T(y) + \lambda \int_{-a}^{a} f[\psi_n(z)]K(y,z)dz \qquad (n = 0, 1, 2, \ldots) \tag{21}$$

with $\psi_0(y) \equiv 0$. It is to be noted that $\psi \equiv 0$ corresponds to a step function with

$$v(y) = v_1 \qquad \text{for} \quad y < 0$$

and

$$v(y) = v_2 \qquad \text{for} \quad y > 0$$

It is seen from the recursion equation (21) that, if the sequence converges, the limit function must be a solution to equation (19). Necessary and sufficient conditions for the convergence of the sequence $\left\{\psi_n(y)\right\}$ are not known. The following sufficient conditions (Hammerstein, ref. 4), are quite strong in restricting the magnitudes of λ, $f[\psi(y)]$, and $K(y,z)$. They are:

(i) $A^2(y) = \lambda^2 \int_{-a}^{a} K^2(y,z)dz$ exists

(ii) $f[\psi(y)]$ satisfies a Lipschitz condition

$$|f(\psi_1) - f(\psi_2)| < C(y)|\psi_1 - \psi_2|$$

(iii) where $\int_{-a}^{a} A^2(y)C^2(y)dy = M^2 < 1$

Notice that (ii) does not restrict $C(y)$ to be continuous; hence, $f(\psi)$ need not be continuous. In fact, all that is required of any of the functions is that their squares be integrable. (Weaker and more precise conditions are given by Hammerstein.) Certainly, all functions of the governing equation (19) have integrable squares. Condition (i) is satisfied for any kernel

$$K(y,z) = E_n(|y - z|) \qquad (n > 1)$$

or

$$K(y,z) = \mathrm{sgn}(y - z)E_n(|y - z|)$$

These are the kernel functions encountered in many radiation problems.

The most stringent of the conditions for convergence is condition (iii), which restricts the magnitudes of λ, $K(y,z)$, and $f[\psi(y)]$. It is easily shown that $|K(y,z)| \leq 1$ and $f[\psi(y)] \leq 1/4^4$ in equation (19), but the parameter λ, for a given gas, is a function of both shock strength and radiation intensity (ref. 1). When both are "sufficiently strong," it was found by numerical experiment that the sequence does not converge.

To see that the sequence $\left\{\psi_n(y)\right\}$ must converge when (i), (ii), and (iii) are satisfied, it is of value to note that (21) gives

$$\psi_{n+1}(y) - \psi_n(y) = \lambda\int_{-a}^{a}\left\{f[\psi_n(z)] - f[\psi_{n-1}(z)]\right\}K(y,z)dz \tag{22}$$

Using (ii), we have

$$|\psi_{n+1}(y) - \psi_n(y)| < \lambda\int_{-a}^{a}C(z)|\psi_n(z) - \psi_{n-1}(z)||K(y,z)|dz \tag{23}$$

By use of the Cauchy-Bunyakovski-Schwarz inequality, equation (23) gives

$$\begin{aligned}|\psi_{n+1}(y) - \psi_n(y)|^2 &< \lambda^2\int_{-a}^{a}K^2(y,z)dz\int_{-a}^{a}C^2(z)[\psi_n(z) - \psi_{n-1}(z)]^2\,dz \\ &= A^2(y)\int_{-a}^{a}C^2(z)[\psi_n(z) - \psi_{n-1}(z)]^2\,dz\end{aligned} \tag{24}$$

We have

$$\psi_1{}^2(z) = \left[\lambda\int_{-a}^{a}f(0)K(y,z)dz\right]^2 \leq k^2A^2(y)$$

with

$$k^2 = \int_{-a}^{a}f^2(0)dz$$

Then from (iii) (24) gives

$$[\psi_2(y) - \psi_1(y)]^2 < k^2A^2(y)\int_{-a}^{a} C^2(z)A^2(z)dz < k^2M^2A^2(y)$$

Continuing

$$[\psi_3(y) - \psi_2(y)]^2 < k^2M^4A^2(y)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$[\psi_{n+1}(y) - \psi_n(y)]^2 < k^2M^{2n}A^2(y)$$

and so

$$|\psi_{n+1}(y) - \psi_n(y)| < kM^n|A(y)| \tag{25}$$

This last equation shows convergence of the sequence $\left\{\psi_n(y)\right\}$ at every point where $|A(y)| < \infty$, since (iii) assumes $M^2 < 1$.

It has been stated that for a given gas, the parameter λ is a function of shock strength and radiation intensity. To see this, it is necessary to recall that

$$\lambda = \frac{4(\gamma - 1)\sigma c_1^6}{\Gamma(\gamma + 1)R^4}$$

while, from equations (5) and (9)

$$q = \frac{2\sigma c_1^8}{R^4}\int_{-\infty}^{\infty}[v(z) - v^2(z)]^4K(y,z)dz$$

so that

$$\lambda = \frac{2q}{\Gamma(\gamma + 1)c_1^2\int_{-\infty}^{\infty}[v(z) - v^2(z)]^4K(y,z)dz}$$

This shows that λ is a function of q, the radiative heat flux and, hence, is a function of radiation intensity. That λ is also a function of shock strength is evident from equation (11) which shows that λ is a function of v_1 and v_2 which, in turn, determine shock strength.

Numerical Procedure

Construction of the sequence $\left\{\psi_n(y)\right\}$ for the governing equation (19) was accomplished by an iterative procedure on an IBM 7090 computer. The integration was carried out using Gaussian quadrature methods with the number of abscissas increasing as $y = 0$ is approached.

With $\psi_0(y) \equiv 0$, it is possible to obtain an analytic expression for $\psi_1(y)$ from equation (21). The number and position of the abscissas were determined by the requirement that the analytic and numerical methods yield values for $v[\psi_1(y)]$ which agreed at every abscissa to within 10^{-5}.

With the constant a set equal to 5, the error in computing $v(y)$ from equation (11) is less than $2/4^4 \times 10^{-3} \sim 8 \times 10^{-6}$. Since this error is the same order of magnitude as that in the numerical integration, the interval [-5,5] was chosen for the range of the integral in equation (11).

The interval [-5,5] was subdivided by the insertion of a total of 129 points, $y_j (j = 1, 2, \ldots, 129)$. The values of $\psi(y) = \lim\limits_{n \to \infty} \psi_n(y)$ were approximated at these points, the approximation being considered accurate when

$$\sum_{j=1}^{j=129} \left| v[\psi_n(y_j)] - v[\psi_{n-1}(y_j)] \right| < 10^{-4}$$

An exponential interpolation was used to determine values of $\psi_n(y)$ between the chosen abscissas.

RESULTS AND DISCUSSION

Figure 1 depicts two velocity profiles which result from dissimilar physical conditions. The data for both graphs were obtained by the methods of this paper with particular values of λ chosen to coincide with two examples considered in reference 1. In order that the profiles of figure 1 could be accurately compared with their counterparts from reference 1, the numerical values used in constructing the graphs of reference 1 were obtained. Comparison was then made between numerical values from reference 1 and those from the direct solution, used in constructing figure 1. The two methods yield results which differ so slightly that graphical comparison is best made on the basis of percentage error.

$$\text{Percentage error} = \frac{(v(y) \text{ from reference 1}) - (v(y) \text{ from direct solution})}{v(y) \text{ from direct solution}}$$

The comparison is made in figure 2. It is to be noticed that the close agreement persists as λ changes by a factor of 10, as it does when the physical situation changes from a weak shock and weak radiation to a strong shock and strong radiation. The largest percentage error in the entire interval $[-5,5]$, for any of the examples compared, was 1.07 percent; this was obtained with λ equal to 100. For $|y| > 5$, both the direct solution and the results of reference 1 give the same result to 6 decimal places.

It will be noticed that figure 2 shows the percentage error to be nearly symmetrical about the shock for $\lambda \sim 10$, but not so symmetrical for $\lambda \sim 100$. There is no reason to believe that the difference between the results from the two methods should exhibit symmetry. Both methods did, however, clearly show the velocity profile to be asymmetrical about the shock, with the asymmetry becoming more noticeable with increasing λ.

It is not known what part of the actual difference in results is due to the errors inherent in numerical processes. As stated earlier, there is reason to believe that the direct solutions by methods of this paper yield velocity profiles which are accurate to at least 10^{-4} in $v(y)$.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Sept. 26, 1963

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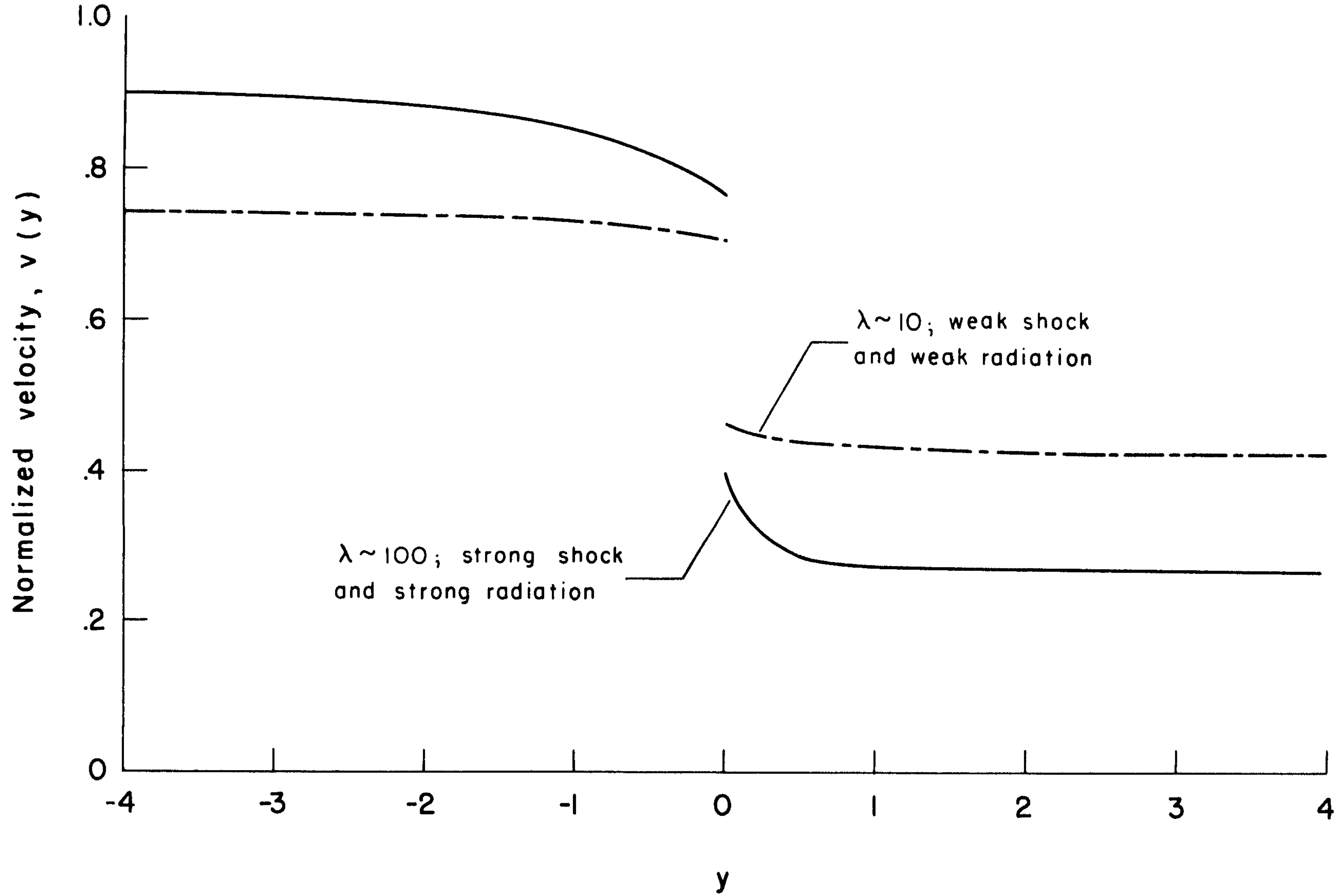

Figure 1.- Velocity profiles for two radiation resisted shock waves.

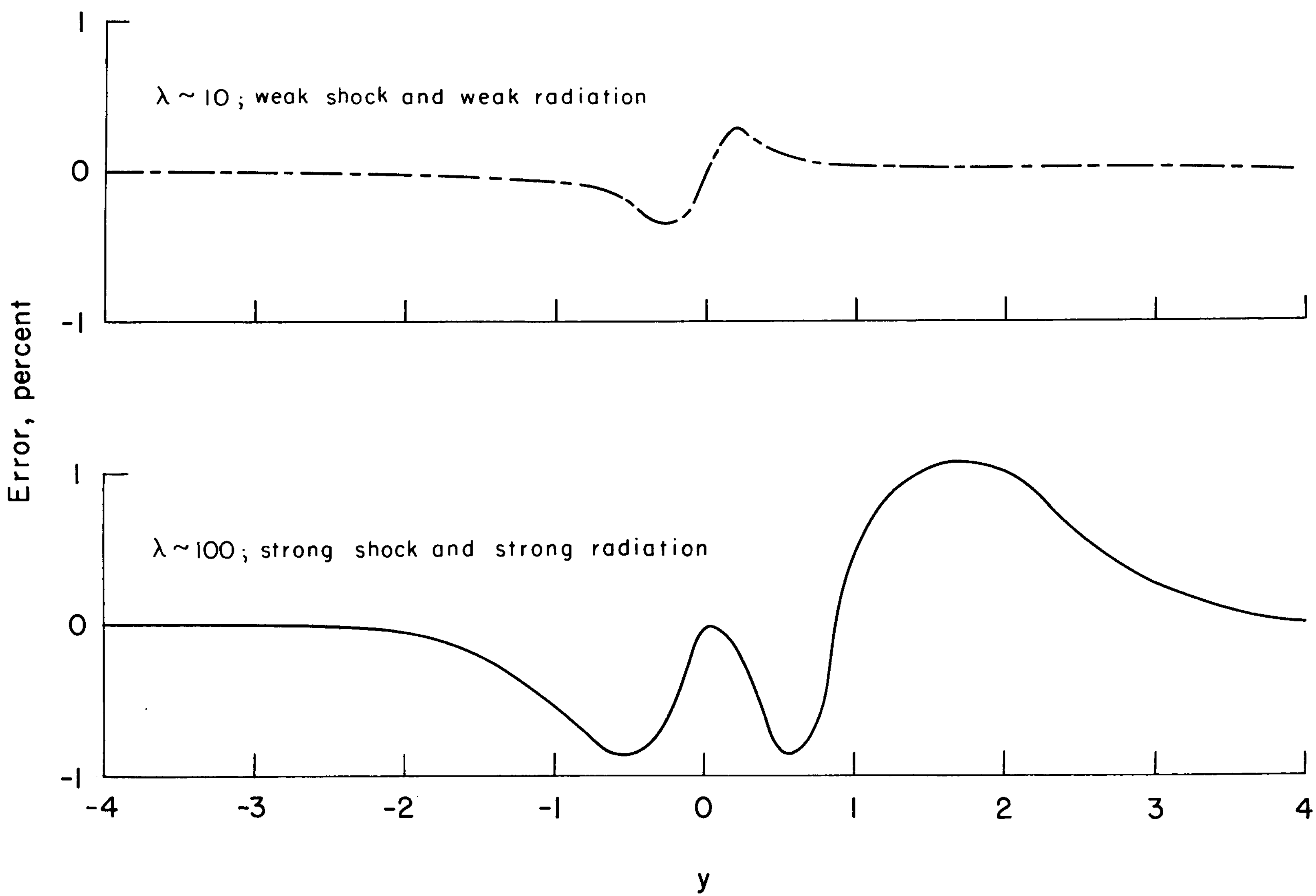

Figure 2.- Percentage error introduced by the approximation $E_2(t) \approx me^{-nt}$.

NASA-Langley, 1963 A-854